

# Supersymmetric classical cosmology

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In this work a supersymmetric cosmological model is analyzed in which we consider a general superfield action of a homogeneous scalar field supermultiplet interacting with the scale factor in a supersymmetric FRW model. There appear fermionic superpartners associated with both the scale factor and the scalar field, and classical equations of motion are obtained from the super-Wheeler-DeWitt equation through the usual WKB method. The resulting supersymmetric Einstein-Klein-Gordon equations contain extra radiation and stiff matter terms, and we study their solutions in flat space for different scalar field potentials. The solutions are compared to the standard case, in particular those corresponding to the exponential potential, and their implications for the dynamics of the early Universe are discussed in turn.

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## I. INTRODUCTION

From the latest observations, we do know that about 95% of matter in the Universe [1] is of non-baryonic nature, and the rest is constituted by radiation, baryons, neutrinos, and all other particles we understand well in the Standard Model of Particle Physics. The most successful models until now is the so called Lambda Cold Dark Matter ( $\Lambda$ CDM) model[2–7], which is able to explain and to fit reasonably well all cosmological observations.

In the last decades cosmologists have made use of scalar fields in the description of various aspects of cosmology. They are known in models of inflation, and more recently in models of dark energy, see[8, 9] and references therein. But scalar fields have been considered too for models of dark matter[10, 11].

This flexibility of scalar field models to describe different phenomena comes from the properties of the self-interacting scalar field potential  $V(\varphi)$  that is specified for each model. Currently, there is no underlying principle that uniquely specifies the potential for the scalar field and many proposals have been considered[8, 12, 13]. Some were based in new particle physics and gravitational theories, other were postulated ad-hoc to obtain the desired evolution.

On the other hand, the physics required to understand

the early Universe should be necessarily rooted in a theory of quantum gravity. Furthermore, it would probably be adequate to consider scenarios where both bosonic and fermionic matter fields would be present on an equal footing.

In considering the quantum creation of the Universe we are of course dealing with the earliest epochs of the Universe's existence, at which time it is believed that supersymmetry would not yet be broken. The inclusion of supersymmetry could therefore be vital from the point of view of physical consistency.

For these and other physical reasons supersymmetric quantum cosmology emerged as an active area of research. The first model proposed[14] was based on the fact that, shortly after the invention of supergravity, it was shown[15, 16] that this theory provides a natural classical square root equations and their corresponding Hamiltonians.

A second method later proposed was a superfield formulation, in which is possible to obtain the corresponding fermionic partners and also being able to incorporate matter in a simpler way[17–19]. A third method allows us to define a *square root* of the potential, in the minisuperspace, of the cosmological model of interest and consequently operators which square results is the Hamiltonian[20–22].

So, in the same way that we seek a desirable scalar field potential to explain the evolution (and early times) of the Universe from the point of view of standard General Relativity, we can reconcile these requirements along with the ideas of local supersymmetry using now *superpotentials*. For this purpose we need to model a super-

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symmetric quantum cosmological landscape and see what happens now with the super expansion factor and with the super-scalar fields. It is then important to find the influence of the "fermionic" variables in these superfields and how they would alter the dynamics of usual cosmological models.

In this work, we consider a Hamiltonian for a homogeneous super scale factor, which is a supermultiplet (with four components and different signs) in supergravity  $N = 2$ , and that interacts with a super scalar field (also a supermultiplet)[19]. We shall promote this Hamiltonian to be an operator, representing the Grassmann variables by matrices, then by means of the WKB procedure we find two (independent) classical evolution equations. Those associated with the scalar field are obtained through Hamilton equations.

This procedure gives us a modified Einstein-Klein-Gordon (EKG) set of equations (that we call SUSY-EKG equations) due to the indirect presence of the "gravitinos" and the "fermionic" variables corresponding to the scalar field, which are inherently contained in each entry of the supermultiplets. From a phenomenological point of view, the new extra terms in the model offer different kind of components that behave as radiation and stiff matter.

The paper is organized as follows. In Sec. II, we outline the procedure that allows us to define the superfields associated with the expansion factor and the scalar field so that we can generalize the usual action of Cosmology. We find the Hamiltonian of the system, which already contains extra terms depending upon the Grassmanian variables associated with the scale factor and the scalar field.

The Grassmanian variables are represented as matrices, and then the Hamiltonian operator is a matrix itself with four components. We focus our attention in its two independent components, and apply to them the usual WKB method to get *classical* equations of motion. Some solutions are found for the cases of a free scalar field and of a constant scalar field potential which is negative definite.

Sec. III is dedicated to the analysis of the case in which the scalar field is endowed with an exponential super potential. We first show that there is an exact scaling solution, in which all energy terms behave like stiff matter. To have a complete picture of the solutions, the equations of motion are written as a dynamical system, and we study its critical points and general trajectories in the phase space of the resulting variables.

Finally, Sec. IV is devoted to conclusions and comments about the general properties of the SUSY-EKG equations and their solutions.

## II. THE SUSY-EKG EQUATIONS FOR A FRW UNIVERSE

In this section, we describe the main features of SGR cosmology, and for this we will write the supersymmetric version of the WDW equation according to the superfield method outlined in the introduction.

### A. Mathematical background

For a homogeneous and isotropic universe, we write the Friedmann-Robertson-Walker metric as (in units with  $c = 1$ ),

$$ds^2 = -N(t)dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (1)$$

where  $a(t)$  is the (time-dependent) scale factor,  $N(t)$  is the lapse function, and  $k$  is the curvature constant. Then, we can write the total action representing a (real) scalar field  $\phi$  endowed with a scalar field potential  $V(\phi)$ , and interacting with the expansion factor as

$$S = \frac{6}{8\pi G} \int \left( -\frac{a\dot{a}^2}{2N} + \frac{1}{2}kNa \right) dt + S_{mat}(\Phi). \quad (2)$$

The equations of motion arising from this action, for  $N = 1$ , are

$$\dot{H} = -\frac{\kappa^2}{6}\dot{\phi}^2, \quad (3a)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV}{d\phi}, \quad (3b)$$

together with the (constraint) Friedmann equation

$$H^2 = \frac{\kappa^2}{3} \left( \dot{\phi}^2 + V(\phi) \right), \quad (4)$$

where  $H \equiv \dot{a}/a$  is the Hubble parameter, and  $\kappa^2 = 8\pi G$ . Eqs. (3a), (3b), and (4) are the representative equations of motion of a FRW universe driven by a scalar field.

We want now to review the procedure that one of us and collaborators have followed to construct a superfield action for the FRW model interacting with a (homogeneous) scalar supermultiplet[19], and from this the superhamiltonian associated with it.

The most general superfield action[17, 18, 23] has the form

$$S = \int 6 \left[ -\frac{1}{2\kappa^2} \frac{\mathcal{A}}{\mathcal{N}} \mathcal{D}_{\bar{\eta}} \mathcal{A} \mathcal{D}_{\eta} \mathcal{A} + \frac{\sqrt{k}}{2\kappa^2} \mathcal{A}^2 \right] d\eta d\bar{\eta} dt + \int \left[ \frac{1}{2} \frac{\mathcal{A}^3}{\mathcal{N}} \mathcal{D}_{\bar{\eta}} \Phi \mathcal{D}_{\eta} \Phi - 2\mathcal{A}^3 g(\Phi) \right] d\eta d\bar{\eta} dt, \quad (5)$$

where  $k = 0, 1$  denotes flat and closed space, and  $\kappa^2 = 8\pi G_N$ , where  $G_N$  is Newton's gravitational constant. The units for the constants and fields in this work are

the following:  $[\kappa^2] = \ell^2$ ,  $[\mathcal{N}] = \ell^0$ ,  $[\mathcal{A}] = \ell^1$ ,  $[\Phi] = \ell^{-1}$ ,  $[g(\Phi)] = \ell^{-3}$ , where  $\ell$  corresponds to units of length. Besides,  $\mathcal{D}_\eta = \partial_\eta + i\bar{\eta}\partial_t$  and  $\mathcal{D}_{\bar{\eta}} = -\partial_{\bar{\eta}} - i\eta\partial_t$  are the supercovariant derivatives of the conformal supersymmetry  $N = 2$ , which has dimension  $[\mathcal{D}_\eta] = [\mathcal{D}_{\bar{\eta}}] = \ell^{-1/2}$ .

For the one-dimensional grassmann superfield  $\mathcal{N}(t, \eta, \bar{\eta})$  ( $\mathcal{N} = \mathcal{N}^\dagger$ ), we have the following series expansion,

$$\mathcal{N}(t, \eta, \bar{\eta}) = N(t) + i\eta\bar{\psi}'(t) + i\bar{\eta}\psi'(t) + \eta\bar{\eta}\mathcal{V}', \quad (6)$$

where  $N(t)$  is the lapse function, and we have also introduced the reparametrization  $\psi'(t) = N^{1/2}(t)\psi(t)$ , and  $\mathcal{V}'(t) = \mathcal{N}(t)\mathcal{V}(t) + \bar{\psi}(t)\psi(t)$ . The Taylor series expansion of the superfield  $\mathcal{A}$  has a similar form,

$$\mathcal{A}(t, \eta, \bar{\eta}) = a(t) + i\eta\bar{\lambda}'(t) + i\bar{\eta}\lambda'(t) + \eta\bar{\eta}\mathcal{B}', \quad (7)$$

where  $a(t)$  is the scale factor,  $\lambda'(t) = \kappa N^{1/2}(t)\lambda(t)$ , and  $\mathcal{B}'(t) = \kappa N(t)\mathcal{B}(t) + (1/2)\kappa(\bar{\psi}(t)\lambda(t) - \psi(t)\bar{\lambda}(t))$ . Likewise, the scalar superfield  $\Phi(t, \eta, \bar{\eta})$  may be written as ( $\Phi = \Phi^\dagger$ ),

$$\Phi(t, \eta, \bar{\eta}) = \phi(t) + i\eta\bar{\chi}'(t) + i\bar{\eta}\chi'(t) + \eta\bar{\eta}\mathcal{F}', \quad (8)$$

where  $\chi'(t) = N^{1/2}(t)\chi(t)$ , and  $\mathcal{F}'(t) = N(t)F(t) + (1/2)(\bar{\psi}(t)\chi(t) - \psi(t)\bar{\chi}(t))$ .

As it was shown in Ref.[17], we now expand the action (5) in terms of the superfield components (6), (7), and (8), and integrate over the Grassmann complex coordinates  $\eta$  and  $\bar{\eta}$ . If we redefine

$$\lambda(t) \rightarrow \frac{1}{3}a^{-1/2}(t)\lambda(t), \quad \chi(t) \rightarrow a^{-3/2}(t)\chi(t), \quad (9)$$

it is possible to find the Lagrangian, and from it the superHamiltonian can be constructed, namely,

$$\begin{aligned} \mathcal{H} = & -\frac{\kappa^2}{12}a^{1/2}\Pi_a a^{1/2}\Pi_a - \frac{3ka}{\kappa^2} - \frac{1}{6}\frac{\sqrt{k}}{a}[\bar{\lambda}, \lambda] + \frac{\Pi_\varphi^2}{2a^3} \\ & - \frac{i\kappa}{4a^3}\Pi_\varphi([\bar{\lambda}, \chi] + [\lambda, \bar{\chi}]) - \frac{\kappa^2}{16a^3}[\bar{\lambda}, \lambda][\bar{\chi}, \chi] \\ & + \frac{3\sqrt{k}}{4a}[\bar{\chi}, \chi] + \frac{\kappa^2}{2}g(\varphi)[\bar{\lambda}, \lambda] + 6\sqrt{k}g(\varphi)a^2 \\ & + a^3V(\varphi) + \frac{3}{4}\kappa^2g(\varphi)[\bar{\chi}, \chi] + \frac{\partial^2 g(\varphi)}{\partial\varphi^2}[\bar{\chi}, \chi] \\ & + \frac{\kappa}{2}\frac{\partial g(\varphi)}{\partial\varphi}([\bar{\lambda}, \chi] - [\lambda, \bar{\chi}]). \end{aligned} \quad (10)$$

where the scalar field potential reads

$$V(\varphi) = 2\left(\frac{\partial g(\varphi)}{\partial\varphi}\right)^2 - 3\kappa^2g^2(\varphi). \quad (11)$$

Notice that, in general, the scalar potential (11) is not positive semi-definite. The relevant term in Eq. (11) is  $g(\varphi)$ , which is related to the superpotential and whose form shall be chosen appropriately for the cosmological model under study.

In the quantum (canonical) formalism the Grassmannian variables  $\lambda$ ,  $\bar{\lambda}$ ,  $\chi$ , and  $\bar{\chi}$ , — by the anticommutators as

$$\{\lambda, \bar{\lambda}\} = -\frac{3}{2}, \quad \{\chi, \bar{\chi}\} = 1, \quad (12)$$

and they can be considered as generators of the Clifford algebra, as well as the commutators

$$[a, \Pi_a] = -i, \quad [\phi, \Pi_\phi] = -i. \quad (13)$$

We can choose a matrix representation for the "fermionic" operators  $\lambda$ ,  $\bar{\lambda}$ ,  $\chi$ , and  $\bar{\chi}$ , in the form of a tensorial products of  $2 \times 2$  matrices,

$$\lambda = \sqrt{\frac{3}{2}}\sigma_- \otimes 1, \quad \bar{\lambda} = -\sqrt{\frac{3}{2}}\sigma_+ \otimes 1, \quad (14a)$$

$$\chi = \sigma_3 \otimes \sigma_-, \quad \bar{\chi} = \sigma_3 \otimes \sigma_+, \quad (14b)$$

where  $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ ,  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are Pauli matrices.

## B. The classical landscape

As we have already mentioned in the introduction, our objective is now to construct the classical equations that corresponds to the Hamiltonian (10).

First, we promote it to an operator  $\hat{\mathcal{H}}$  by realizing the "fermionic" variables as the matrices (14) and, as usual,  $\Pi_a = i\partial_a$  and  $\Pi_\phi = i\partial_\phi$ . By these means, we will get a quantum Hamiltonian operator that should fulfill  $\hat{\mathcal{H}}|\Psi\rangle = 0$ . Because the matrices in (14) are  $4 \times 4$ , the wave function  $\Psi$  will have four components.

It can be shown[19] that the components  $\Psi_1$  and  $\Psi_4$  satisfy independent equations, whereas  $\Psi_2$  and  $\Psi_3$  appear coupled in the other two differential equations. In this work we focus our attention in the former case, and apply separately to  $\Psi_1$  and  $\Psi_4$  the WKB method, so that

$$\Psi = e^{(S_a + S_\varphi)}. \quad (15)$$

With this we shall find the classical equations of motion associated to the components  $\Psi_1$  and  $\Psi_4$ , which also correspond to the classical Hamiltonian; from this, classical equations can be obtained for the scalar field  $\varphi$ . Thus, the classical equations of motion are

$$\begin{aligned} \ddot{\varphi} = & -3H\dot{\varphi} - \frac{\partial V}{\partial\varphi} \pm \frac{3}{2}\frac{\kappa^2}{a^3}\frac{\partial g(\varphi)}{\partial\varphi} - 6\frac{\sqrt{k}}{a}\frac{\partial g(\varphi)}{\partial\varphi} \\ & \mp \frac{1}{a^3}\frac{\partial^3 g(\varphi)}{\partial\varphi^3}, \end{aligned} \quad (16a)$$

$$\begin{aligned} H^2 = & \frac{\kappa^2\dot{\varphi}^2}{6} + \frac{\kappa^2}{3}V(\varphi) - \frac{k}{a^2} \pm \frac{\kappa^2\sqrt{k}}{3a^4} + \frac{\kappa^4}{32a^6} \\ & \mp \frac{\kappa^4}{2a^3}g(\varphi) + 2\frac{\kappa^2\sqrt{k}}{a}g(\varphi) \pm \frac{\kappa^2}{3a^3}\frac{\partial^2 g(\varphi)}{\partial\varphi^2}. \end{aligned} \quad (16b)$$

The upper (lower) sign in Eqs. (16) corresponds to the (quantum) equation for  $\Psi_1$  ( $\Psi_4$ ).

We have now SUSY-EKG classical equations that, due to the presence of variables  $\lambda$  and  $\bar{\lambda}$  associated with the "gravitino" and the "fermionic" variables  $\chi$  and  $\bar{\chi}$  associated with the scalar field, considerably differ from the standard EKG equations (3) of General Relativity.

There are extra terms, due to supersymmetry, behaving like radiation ( $a^{-4}$ ) and stiff matter ( $a^{-6}$ ), which should be dominant at very early times, whereas other terms show a combination of scale factor powers mediated by the presence of the superpotential  $g(\phi)$  and its derivatives.

### C. Simple classical examples

As a first instance of a solution, we will consider the case  $g(\varphi) = 0$ , which also corresponds to a null potential,  $V(\varphi) = 0$ . We also set the curvature term  $k = 0$ . The solutions of Eqs. (16) are

$$\dot{\varphi}(t) = \dot{\varphi}_0(a_0/a)^3, \quad (17)$$

$$a^3(t) = a_0^3 + 3 \left( \frac{\kappa^2 \dot{\varphi}_0^2 a_0^6}{6} + \frac{\kappa^4}{32} \right)^{1/2} (t - t_0), \quad (18)$$

where  $t_0$ ,  $\dot{\varphi}_0$ , and  $a_0$  are integration constants. The whole solution corresponds to stiff matter, and is practically the same as in the standard case because the  $g$ -terms disappear from everywhere.

Another more interesting case is that with a constant superpotential,  $g(\varphi) = g_0$ , that corresponds to a constant and negative definite scalar field potential,  $V = -3\kappa^2 g_0^2$ . The scalar field potential is then an effective cosmological constant which is negative definite.

As in the previous case of the free scalar field, the case is simplified because the derivatives of the superpotential disappear from the SUSY-KG equation (16a), though not from the SUSY-Friedmann equation (16b). The cosmological solutions can be expressed as

$$\dot{\varphi}(t) = \dot{\varphi}_0(a_0/a)^3, \quad (19a)$$

$$a^3(t) = \frac{a_0}{4g_0} \left[ \left( \sqrt{\frac{8\dot{\varphi}_0^2 a_0^6}{3\kappa^2} + \frac{9}{8}} \right) \sin[3\kappa^2 g_0(t - t_0)] \mp \frac{19}{8} \right], \quad (19b)$$

where again  $\dot{\varphi}_0$  and  $a_0$  are integration constants.

Because the scale factor is a positive quantity, the only acceptable solution is when the amplitude of the sinus function is less or equal to one. It is then clear that the scale factor has a periodic solution in which  $a_0$  is the amplitude at maximum expansion. In other words, Eq. (19b) represents an oscillatory Universe.

## III. SUSY COSMOLOGY WITH AN EXPONENTIAL SUPERPOTENTIAL

The possible cosmological roles of exponential potentials in scalar field models have been thoroughly

investigated in the specialized literature [24–36], see also [8, 9, 13], almost always as a means for driving a period of cosmological inflation, but also as possible candidates for dark matter and dark energy.

Scalar field cosmologies with an exponential potentials are, as compared to others, mathematically simple, and their solutions have many interesting features. For the purposes of this work, we only mention the possibility of having inflationary solutions and the appearance of the so-called scaling solutions, which are nicely illustrated in, for instance, the dynamical system study presented in [32].

The inflationary solution for exponential potential is the simple power law inflation [13], which however never ends and needs modifications to provide a graceful exit towards a Hot Big Bang model. On the other hand, the scaling solution arises whenever the scalar field is accompanied by another matter fields, so that both fields evolve with a fixed ratio of their energy densities, see for instance [26, 27, 29–32, 36].

In this section we explore in detail the type of solutions permitted by our (classical) SUSY cosmological model when the scalar field is endowed with an exponential potential. Our main interest will be to find inflationary and scaling solutions. Even though we are not considering extra matter fields apart from the scalar field, the new terms in Eqs. (16) will play the role of companion fields which should impose a non-trivial behaviour upon the field  $\varphi$ .

Let us consider the following superpotential and potential, respectively,

$$g(\varphi) = g_0 e^{-\lambda\kappa\varphi/2}, \quad (20a)$$

$$V(\varphi) = V_0 e^{-\lambda\kappa\varphi}, \quad V_0 \equiv \frac{\kappa^2 g_0^2}{2} (\lambda^2 - 6), \quad (20b)$$

where the potential parameters were chosen to ease their comparison with the standard case; notice that in order to avoid a negative definite potential we should impose the condition  $\lambda > \sqrt{6}$ . The equations of motion (16) with an exponential superpotential explicitly read

$$\ddot{\varphi} = -3\frac{\dot{a}}{a}\dot{\varphi} + \lambda\kappa V \pm (\lambda^2 - 6)\frac{\lambda\kappa^3 g}{8a^3}, \quad (21a)$$

$$H^2 = \frac{\kappa^2}{6}\dot{\varphi}^2 + \frac{\kappa^2}{3}V + \frac{\kappa^4}{32a^6} \pm (\lambda^2 - 6)\frac{\kappa^4 g}{12a^3}. \quad (21b)$$

### A. Exact SUSY scaling solution

We present here a first (exact) solution of Eqs. (21) that we shall call *scaling solution*, because of its resemblance with the scaling behavior exponential potentials show in standard cosmology [29, 33, 34].

It can be noticed that there is a stiff matter term in Eq. (21b), and that the superpotential  $g$  appears accompanied by factor  $a^{-3}$ . Thus, one can foresee that there must be a stiff matter solution of the equations of motion, so that  $a \sim t^{1/3}$ , as long as  $g \sim a^{-3}$  and  $V \sim a^{-6}$ .

It can be shown, just by direct substitution in Eqs. (21), that the exact scaling solution is

$$a(t) = a_0(t/t_0)^{1/3}, \quad (22a)$$

$$\kappa\varphi(t) = \frac{2}{\lambda} \ln(t/t_0), \quad (22b)$$

$$g(t) = g_0(t_0/t), \quad g_0 = \mp a_0^{-3}, \quad (22c)$$

where  $a_0$  is an appropriate constant to accomplish Eqs. (21)[37].

The scaling solution corresponds to stiff fluid matter, as revealed by the power law behaviour of the scale factor in Eq. (22a); this is probably not surprising, because we have already noticed the presence of a stiff-term in the SUSY Friedmann equation (16b). This solution is not inflationary, but its existence indicates its possible importance in the early dynamics of the models considered here.

## B. Dynamical System Structure

Our next step is to study the evolution of our SUSY classical model in which the scalar field  $\varphi$  is endowed with an exponential potential. As in the standard case, it is possible to perform a dynamical study of the cosmological model so that its physically relevant solutions are easily unveiled.

In order to construct a dynamical system for our cosmological model, we follow Ref.[29], see also[31, 34, 36, 38]. One first step is to introduce a set of conveniently chosen variables which may allow the rewriting of the evolution equations as an autonomous phase system subject to a constraint arising from the Friedmann equation.

We choose the following variables,

$$x \equiv \frac{\kappa\dot{\varphi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \quad z \equiv \frac{\kappa^2}{\sqrt{32}a^3H}, \quad (23)$$

which render the Friedmann equation as

$$F(x, y, z) := x^2 + y^2 + z^2 \pm 2\sqrt{(\lambda^2 - 6)/3}yz = 1. \quad (24)$$

The constraint equation (24) follows from Eq. (21b), and we see that variable  $z$  plays the role of an extra fluid term which, contrary to the standard case, see Ref.[29], is not trivially coupled to the scalar field variables.

We shall restrict ourselves to the part of the phase space that makes physical sense, and this is the range  $0 \leq |x|, y, z < \infty$ , which is the part that corresponds to expanding universes only. Combining expressions (21) and (23), the equations of motion read

$$x' = -3x - \frac{\dot{H}}{H^2}x + \sqrt{\frac{3}{2}}\lambda y^2 \pm \frac{\lambda\sqrt{\lambda^2 - 6}}{\sqrt{2}}yz, \quad (25a)$$

$$y' = \sqrt{\frac{3}{2}}\lambda xy - \frac{\dot{H}}{H^2}y, \quad (25b)$$

$$z' = -3z - \frac{\dot{H}}{H^2}z, \quad (25c)$$

where

$$\frac{\dot{H}}{H^2} = -3x^2 - 3z^2 \mp \sqrt{3(\lambda^2 - 6)}yz. \quad (26)$$

Here primes denote derivative with respect to the logarithm of the scale factor,  $N = \ln(a)$ . The evolution of phase space variables  $x$ ,  $y$ , and  $z$  takes place only on the constraint surface described by Eq. (24).

Notice that there is a symmetry in Eqs. (25) with respect to the double sign ( $\pm$ ) and variable  $y$ . The case with sign ( $-$ ) can be obtained from the case with ( $+$ ) if we change  $y \rightarrow -y$ ; and vice versa. As we shall see below, we will only study the case with the lower signs since only for them is that we can obtain positive results for  $y$ .

We show in Table I the critical points of the dynamical system (25) and their (linear) stability properties.

There are five critical points, in close similarity to the standard case, whose main features are described next.

- Stiff matter domination. The potential variable is null,  $y = 0$ , and then the dynamical system is equivalent to the standard case of stiff fluid matter ( $a^{-6}$ ) plus a free scalar field ( $\dot{\phi} \sim a^{-3}$ ), so that  $x^2 + z^2 = 1$ . Particular (well known) cases are:
  - Point A, stiff fluid domination. The scalar field variables  $x$  and  $y$  are both null, and then this point represents the complete domination of the stiff matter term,  $z = 1$ . This solution exists also in the standard cosmological case.
  - Points B, kinetic domination. They represent the domination of the scalar field's kinetic energy,  $x = \pm 1$ , and  $y = 0 = z$ . This solution exists also in the standard cosmological case.
  - Points C, joint kinetic and stiff domination. This is a (restricted) scaling solution of stiff nature which is satisfied by all points inside a unitary circle on the plane  $y = 0$ . This set of solutions exists also in the standard cosmological case, but it was missed in the analysis of Ref.[29]: it is the line segment  $x = [-1, 1]$  at  $y = 0$ . Notice that points A (B) can be seen as extreme C-points as  $z \rightarrow 1$  ( $z \rightarrow 0$ ).
- Point D, scalar field domination. It is the coexistence of the (scalar) kinetic and potential energies,  $x^2 + y^2 = 1$ , and then the point is located in the unitary circumference on the plane  $z = 0$ . Notice, however, that the existence of this point requires  $\lambda^2 < 6$ , which is in contradiction with our earlier assumption that  $\lambda^2 > 6$ , see the scalar field potential defined through Eq. (20b).
- Point E, scaling solution. This point corresponds to the scaling solution in Sec. III A, and represents the coexistence of all energy terms in the equations of motion. It should be noticed that, contrary to the present work, in the standard cosmological case the

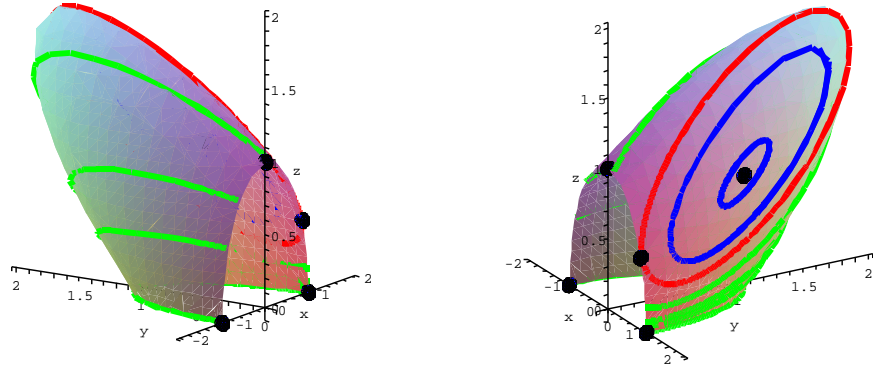


FIG. 1. Different views of the phase space of the dynamical system (25) for the particular value  $\lambda = 2.85$ , so that  $6 < \lambda^2 < 9$ ; the 3-d surface represents the constraint  $F(x, y, z) = 1$ , see Eq. (24), whereas the curves are solutions of the dynamical system for diverse initial conditions. The dots denote the critical points A, B, C and E shown in Table I, and the trajectories reveal their stability properties as described in the text. In particular, notice the largest (red) loop that encircles point E (see figure on the right): it is a homoclinic trajectory that departs from and arrives to the same critical point.

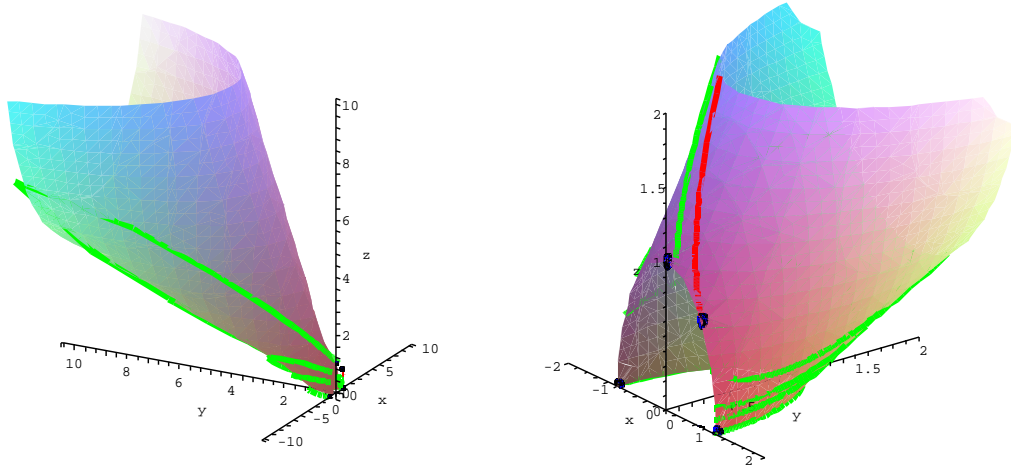


FIG. 2. Different views of the phase space of the dynamical system (25) for the particular value  $\lambda = 3.5$ , so that now  $9 < \lambda^2$ ; the 3-d surface represents the constraint  $F(x, y, z) = 1$ , see Eq. (24), whereas the curves are solutions of the dynamical system for diverse initial conditions. The dots denote the critical points A, B, and C shown in Table I, and the trajectories reveal their stability properties as described in the text. Point E does not exist in this case. As in Fig. 1, there also exists the (red) homoclinic trajectory that departs from and arrives to the same critical point located at  $(\sqrt{6}/\lambda, 0, \sqrt{1 - 6/\lambda^2})$ . However, we were not able to show the complete closed trajectory due to numerical limitations.

scaling solution in the presence of stiff fluid matter necessarily requires  $y = 0$ , see Table I in Ref.[29].

$\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$ , where  $\mathbf{x} = (x, y, z)$  and  $\mathbf{u} = (\delta x, \delta y, \delta z)$ . The equations of motion (25) can be written as  $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ ,

The stability of the points is investigated through linear perturbations around the critical values of the form

TABLE I. Critical points  $(x_*, y_*, z_*)$  of the dynamical system (25) that represents the classical supercosmology of a scalar field endowed with an exponential superpotential. See also Fig. 1 for a graphical representation of the phase space.

Label	$x_*$	$y_*$	$z_*$	Existence	Stability
A	0	0	1	$\forall \lambda$	Unstable
B	-1	0	0	$\forall \lambda$	Unstable
	1	0	0	$\forall \lambda$	Unstable for $\lambda^2 < 6$ Saddle for $\lambda^2 > 6$
C	$-\sqrt{1 - z_*^2}$	0	$z_*$	$\forall \lambda, z_* < 1$	Unstable
	$\sqrt{1 - z_*^2}$	0	$z_*$	$\forall \lambda, z_* < 1$	Unstable for $\sqrt{1 - 6/\lambda^2} < z < 1$ Saddle for $0 < z < \sqrt{1 - 6/\lambda^2}$
D	$\lambda/\sqrt{6}$	$\sqrt{1 - \lambda^2/6}$	0	$\lambda^2 < 6$	Stable node
E	$\sqrt{6}/\lambda$	$(\lambda^2 - 6)/[\lambda\sqrt{9 - \lambda^2}]$	$\sqrt{3(\lambda^2 - 6)}/[\lambda\sqrt{9 - \lambda^2}]$	$6 < \lambda^2 < 9$	Stable centre

which upon linearization reads

$$\mathbf{u}' = \mathcal{M}\mathbf{u}, \quad \mathcal{M}_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}_*}, \quad (27)$$

where  $\mathcal{M}$  is called the linearization matrix. The eigenvalues  $\omega$  of  $\mathcal{M}$  determine the stability of the critical points, whereas the eigenvectors  $\eta$  of  $\mathcal{M}$  determine the principal directions of the perturbations. In general, if  $\text{Re}(\omega) < 0$  ( $\text{Re}(\omega) > 0$ ) the critical point is called stable (unstable).

In principle, we should study the perturbations of the three dynamical variables  $(x, y, z)$ , but we should remember that they are not all independent because they are bond together by the Friedmann constraint (24), and the same happens for their perturbations.

The Friedmann constraint defines a two dimensional surface upon which lie all physically relevant phase space trajectories, and then we will be interested on perturbations lying also on the constraint surface. In other words, perturbations which are perpendicular to the constraint surface should be taken away from the analysis.

We can identify the excluded perturbations by comparing their associated eigenvectors with the gradient of the constraint surface at each critical point,

$$\begin{aligned} \nabla F|_{\mathbf{x}_*} = & x_* \mathbf{i} + \left( y_* \pm \sqrt{(\lambda^2 - 6)/3z_*} \right) \mathbf{j} \\ & + \left( z_* \pm \sqrt{(\lambda^2 - 6)/3y_*} \right) \mathbf{k}. \end{aligned} \quad (28)$$

We will only take into account eigenvalues associated to eigenvectors for which  $\eta \cdot \nabla F|_{\mathbf{x}_*} = 0$ . The stability results are also summarized in Table I.

Here we list the eigenvalues of the stability matrix  $\mathcal{M}$  for each of the critical points, only for the perturbations that are compatible with the Friedmann constraint.

- Point A. It is an unstable point with eigenvalues

$$\omega_1 = 3, \quad \omega_2 = 0. \quad (29)$$

The instability happens only along the eigenvector corresponding to  $\omega_1$ , which points in the positive  $y$ -direction. The second eigenvalue is null, and then the system is indifferent under perturbations along the unitary circumference  $x^2 + z^2 = 1$ .

- Point B. The stability eigenvalues for the cases  $x_* = \mp 1$  are

$$\omega_1 = \sqrt{\frac{3}{2}}(\lambda \pm \sqrt{6}), \quad \omega_2 = 0. \quad (30)$$

Similarly to the case of point A, only the eigenvector corresponding to  $\omega_1$ , which points in the positive  $y$ -direction, gives information about the stability of the critical points. We notice that the point at  $x = -1$  is unstable, whereas that at  $x = 1$  can be unstable or saddle, depending upon the value of  $\lambda$ . Again, the second eigenvalue is null, and then the system is indifferent under perturbations along the unitary circumference  $x^2 + z^2 = 1$ .

- Points C. The eigenvalues corresponding to  $x_* = \mp \sqrt{1 - z_*^2}$  are

$$\omega_1 = 3 \pm \sqrt{\frac{3}{2}}\lambda\sqrt{1 - z_*^2}, \quad \omega_2 = 0. \quad (31)$$

These points have stability properties very similar to points A and B, but stability also depends upon their exact location on the unitary circumference  $x^2 + z^2 = 1$ .

Notice that there is a special point, corresponding to  $z_h = \sqrt{1 - 6/\lambda^2}$  and located at  $(\sqrt{6}/\lambda, 0, \sqrt{1 - 6/\lambda^2})$ , which marks the instability-stability transition of the chain of points C. Because of this, there is a particular trajectory that departs from and also arrives to the point. This is called an homoclinic trajectory, and it is the largest loop that encloses point E, see Figs. 1 and 2.

- Point D. Their stability eigenvalues are

$$\omega_1 = -\frac{1}{2}(6 - \lambda^2), \quad \omega_2 = -\frac{1}{2}(6 - \lambda^2). \quad (32)$$

It is a stable point, whenever it exists, i.e. if the case  $\lambda < \sqrt{6}$  is allowed.

- Point E. Its stability eigenvalues are

$$\omega_1 = i\sqrt{\frac{3}{2}} \frac{(\lambda^2 - 6)^{3/2}}{\sqrt{9 - \lambda^2}}, \quad \omega_2 = -i\sqrt{\frac{3}{2}} \frac{(\lambda^2 - 6)^{3/2}}{\sqrt{9 - \lambda^2}}. \quad (33)$$

The eigenvalues are purely imaginary for the range of existence of the critical point, then it is a stable centre. This is confirmed by the closed trajectories around the critical point in Fig. 1.

The overall conclusion is that only critical points A, B, C and E may coexist together in the phase space, because the (inflationary) point D is excluded by the (positivity) restriction  $\lambda^2 > 6$ . What we observe in Figs. 1 and 2 is that the ultimate fate for trajectories is to move around point E in closed loops, for the case  $6 < \lambda^2 < 9$ , or to reach any of the stable points C, for any  $\lambda^2 > 6$ . All possible solutions in the phase space represent, in general, stiff matter solutions.

#### IV. CONCLUSIONS

In this work we have considered a supersymmetric extension of the action of general relativity for a scalar field interacting with the scale factor of the Universe. For this purpose, we have introduced a superfield formulation in which fermionic degrees of freedom are associated to both the scale factor and to the scalar field.

By realizing the algebra of the fermionic variables and representing them as matrices, we get four equations for four components of the wave function. We focus our attention in two of them that are independent, and apply the WKB method in order to get two *classical* SUSY-cosmological equations. The associated equations of motion for the scalar field are obtained by means of Hamilton's equations.

In these supersymmetric Einstein-Klein-Gordon equations (SUSY-EKG), new contributions arise that behave like stiff matter, and some others in which the usual scalar field terms are modified by functions of the scale factor.

For simplicity, we focused our attention in a flat Universe and were able to find exact solutions of the equations of motion. In one of them the scalar field potential is a negative constant, and then the radius of the Universe is a periodic function. A second exact solution of interest corresponds to the case of an exponential (super)potential. This is a scaling solution corresponding to stiff fluid matter as revealed by the power law behaviour of the scale factor.

We performed an analysis of the dynamical system structure of the SUSY-EKG equations in order to find all relevant physical solutions. Not surprisingly, we found, basically, the same solutions that appear in the standard classical case. However, the stability and existence properties of the solutions were strongly modified by the supersymmetric corrections.

In general, we can say that all solutions show that the scale factor and the scalar field degrees of freedom behave like in the case of stiff matter domination, because the inflationary solution is absent. This is a consequence of the correction terms that appear in the equations of motion, in which the scalar field functions are mediated by stiff-matter terms of the scale factor.

All conclusions are consequences only of the supersymmetric nature of the equations of motion, and the solutions are expected to be relevant in the early stages of the Universe. We cannot foretell the consequences that may arise in the case of a less exact supersymmetry, or even a broken one. But these are possibilities that may modify again the phase structure of the solutions and allow the existence of inflationary solutions. This is research beyond the purposes of the present work that we expect to report elsewhere.

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